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International Journal of Solids and Structures 41 (2004) 6023–6040

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Response of monosymmetric thin-walled Timoshenko beams to random excitations

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Received 9 May 2004

Abstract

A study of the bending–torsion coupled random response of the monosymmetric thin-walled beams subjected to various kinds of concentrated and distributed random excitations is dealt with in this paper. The effects of warping stiffness, shear deformation and rotary inertia are included in the present formulations. The random excitations are assumed to be stationary, ergodic and Gaussian. Analytical expressions for the displacement response of the thin-walled beams are obtained by using normal mode superposition method combined with frequency response function method. The proposed method can produce the accurate solutions for the monosymmetric thin-walled Timoshenko beams or simple structures constructed from such beams. The effects of warping stiffness, shear deformation and rotary inertia on the random response of two appropriately chosen thin-walled beams from the literature are demonstrated and discussed.

Keywords: Thin-walled beam; Timoshenko beam; Bending–torsional coupling; Random response; Normal mode method

1. Introduction

The thin-walled beam members are playing an important role in the design of aerospace, automobile and civil structures such as aircraft wings, turbine blades, decks of bridges and axles of vehicles due to their outstanding properties. Such structures are often subjected to dynamic excitations in complex environmental conditions, in order to ensure that their design is reliable and safe, it is essential for design engineers to evaluate the dynamic characteristics of the thin-walled beams accurately. Thus, an engineer designing such a structure needs to be able to predict its response behavior and be able to easily determine what effects design changes might have on those dynamic response.

It is well known that when the cross-sections of the beams have two symmetric axes, the shear center and the centroid of the cross-sections coincide, and all bending and torsional vibrations are independent of each other, this case represents no coupling at all. Then the classical Bernoulli–Euler and/or the Timoshenko beam theory are valid. However, for a large number of practical beams of thin-walled sections, the centroid

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.05.030

and shear center of the cross-sections are obviously noncoincident, the above assumption is not valid. When the cross-sections of the thin-walled beams have only one symmetrical axis, the bending vibration in the direction of the symmetrical axis is independent of the other vibrations. But the bending vibration in the perpendicular direction of the symmetric axis is coupled with torsional vibration.

Because of the practical importance of the thin-walled beams, the coupled vibration analyses of such problems have inspired continuing research interest in recent years. Many researchers have developed the dynamic response analysis methods for beams having double symmetrical axes and structures composed of this kind of beams (Nagem, 1991; Singh and Abdelnaser, 1993; Chang, 1994). There are also a number of studies dealing with coupled bending–torsional vibration of the thin-walled beams, but the available investigations have been concerned mainly with free vibration characteristics. Small carefully selected studies are mentioned as follows. Bishop and Price (1977) studied the coupled bending–torsional vibration of the Timoshenko beams without the warping stiffness included. Hallauer and Liu (1982) and Friberg (1983) derived the exact dynamic stiffness matrix for a bending–torsion coupled Bernoulli–Euler beam with the warping stiffness ignored. Dokumaci (1987) first derived the exact analytical expressions for the solution of the bending–torsion equations without the warping effect. Banerjee and Williams (1992, 1994) derived the analytical expressions for the coupled bending–torsional dynamic stiffness matrix of a Timoshenko beam excluding the warping stiffness effect. Hashemi and Richard (2000) presented a new dynamic finite element for the bending–torsion coupled Bernoulli–Euler beams with the warping stiffness omitted. Bishop et al. (1989) extended the work of Dokumaci by considering the same equations, but with the inclusion of warping effect. They showed that the warping effect could produce significant changes in the natural frequencies of the vibration. Banerjee et al. (1996) formulated an exact dynamic stiffness matrix for a thin-walled Bernoulli–Euler beam with inclusion of the warping stiffness. Tanaka and Bercin (1999) presented the exact solution for the bending–torsion coupled nonsymmetrical Bernoulli–Euler beams including the warping stiffness. Klausbruckner and Pryputniewicz (1995) theoretically, numerically, and experimentally investigated the vibration of the channel beams. They used a thin-walled beam model that included the effect of warping on the torsional vibration (misleadingly identified as Timoshenko theory no shear deformation or rotary inertia effects were included) for their analytical investigation and a three-dimensional finite element model including shear deformation for their numerical analysis. Friberg (1985) and Leung (1991, 1992) developed the dynamic stiffness matrix of a Vlasov beam with the shear deformation completely ignored. Arpaci et al. (2003) presented an exact analytical method to predict the undamped natural frequencies of beams with thin-walled open cross-sections. The effect of shear deformation is neglected in their formulations, although the effects of warping and rotary inertia are taken into account. Kim et al. (2003a) proposed an improved numerical method for the free vibration and stability analysis of nonsymmetric thin-walled beams based on Vlasov beam theory with shear deformation omitted. The effects of shear deformation and rotary inertia were added to the investigation by Bercin and Tanaka (1997). They showed that for the thin-walled open cross-section beams, the shear deformation and rotary inertia can substantially decrease the natural frequencies of the vibration by as much as 60% in the first mode for a very special case. Kim et al. (2003b) extended their previous work by considering the shear deformation effect.

A literature survey reveals that few studies have considered the response behavior of the thin-walled beams subjected to deterministic or random external excitations. Chen and Tamma (1994) employed the finite element method in conjunction with an implicit-starting unconditionally stable methodology for the dynamic computation of the elastic thin-walled open section structures subjected to deterministic excitations. They employed Vlasov's assumptions and both warping stiffness and rotary inertia were included in the developments. But one important parameter, namely the shear deformation, was not included in the formulations and the paper concentrated attention on the deterministic dynamic response. Eslimy-Isfahany et al. (1996) developed an analytical theory to investigate the response of a bending–torsion coupled beam to deterministic and random excitations by using the normal mode method. The authors assumed that the beam twisted according to the Saint-Venant theory and thus no allowance was made for the warping stiffness of the beam cross-section. Such an assumption could lead to large errors when calculating the dynamic response of a thin-walled open section beam. Also the effects of shear deformation and rotary inertia were not included in the formulations.

To the best of author's knowledge, there is no publication available that incorporates several essential effects simultaneously including bending–torsional coupling, shear deformation, rotary inertia and warping stiffness to the random response analysis of the thin-walled beams. This problem is addressed in this paper. The random vibration of the thin-walled Timoshenko beams with monosymmetrical cross-sections is investigated. The effects due to warping stiffness, shear deformation and rotary inertia on the random response of the thin-walled beams are of interest here. Theoretical expressions for the mean square displacement response of the thin-walled beams subjected to various kinds of concentrated and distributed random excitations having stationary and ergodic properties are obtained by using normal mode method combined with frequency response function method.

2. Free vibration of the thin-walled Timoshenko beams

The structural model used in present study is that of a thin-walled beam with arbitrary monosymmetrical cross-section. For illustrative purpose, considering a uniform and straight open section thin-walled beam with length L, shown in Fig. 1. The present thin-walled beam model incorporates the following features including bending–torsion coupling, transverse shear deformation, rotary inertia and warping stiffness. But the effects of secondary warping and warping inertia are considered to be negligibly small and have been neglected in the present theory. The shear center and centriod of the cross-section are denoted by s and c respectively, which are separated by a distance y_c. In the right handed Cartesian coordinate system in Fig. 1, the x-axis is assumed to coincide with the elastic axis (i.e. loci of the shear center of the cross-section of the thin-walled beam). The bending translation in the z-direction and the torsional rotation about the x-axis of the shear center are denoted by $v(x,t)$ and $\psi(x,t)$ respectively, where x and t denote distance from the origin and time respectively. The rotation of the cross-section due to bending alone is denoted by $\theta(x,t)$. The external excitations acting on the thin-walled beam are represented by a force $f(x,t)$ per unit length that parallel to sz-axis and applied to the shear center together with a torque $m(x,t)$ per unit length about sx-axis respectively.

The damped governing differential equations for the bending–torsion coupled forced vibration of the thin-walled beam, which incorporates shear deformation, rotary inertia and warping stiffness, are expressed as (for details of the derivation, see Appendix A)

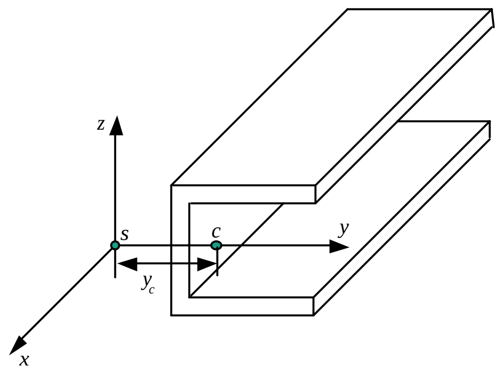

Fig. 1. A uniform straight thin-walled Timoshenko beam.

$$\rho I\ddot{\theta} + c_3\dot{\theta} - EI\theta'' - kAG(v' - \theta) = 0 \tag{1}$$

$$-GJ\psi'' - \mu y_c\ddot{v} + I_s\ddot{\psi} + c_2\dot{\psi} - c_1 y_c\dot{v} + E\Gamma\psi'''' = m(x,t) \tag{2}$$

$$\mu\ddot{v} + c_1(\dot{v} - y_c\dot{\psi}) - \mu y_c\ddot{\psi} - kAG(v'' - \theta') = f(x,t) \tag{3}$$

where E and G are Young's modulus and shear modulus of the thin-walled beam material, respectively. EI, kGA, GJ and $E\Gamma$ are bending stiffness, shear stiffness, torsional stiffness and warping stiffness of the thin-walled beam, respectively. μ is mass of the thin-walled beam per unit length, I is second area moment of inertia of the beam cross-section about y-axis, I_s is polar mass moment of inertia of per unit length thin-walled beam about x-axis, superscript primes and dots denote the derivatives with respect to position x and time t respectively. ρ is the density of the thin-walled beam material, A is the cross-section area of the thin-walled beam, k is the effective area coefficient in shear. The damping coefficients c_1, c_2 and c_3 are the linear viscous damping terms of per unit length thin-walled beam in bending deformation, torsional deformation and rotational deformation due to bending alone respectively.

The exact solutions for the homogeneous equations of motion corresponding to the free vibration are considered first. The external excitations $f(x,t)$ and $m(x,t)$ are set to zero, as are the damping coefficients c_1, c_2 and c_3, in order to determine the natural frequencies and mode shapes of the thin-walled beams. A sinusoidal variation of $v(x,t)$, $\theta(x,t)$ and $\psi(x,t)$ with circular frequency ω_n is assumed to be of the forms

$$v(x,t) = V_n(x)\sin\omega_n t \tag{4}$$

$$\theta(x,t) = \Theta_n(x)\sin\omega_n t \tag{5}$$

$$\psi(x,t) = \Psi_n(x)\sin\omega_n t \tag{6}$$

where $n = 1,2,3,\ldots,$ $V_n(x)$, $\Theta_n(x)$ and $\Psi_n(x)$ are the amplitudes of the sinusoidally varying bending translation $v(x,t)$, bending rotation $\theta(x,t)$ and torsional rotation $\psi(x,t)$ respectively.

Substituting Eqs. (4)–(6) into Eqs. (1)–(3) gives the three simultaneous differential equations for V_n, Θ_n and Ψ_n

$$\rho I\omega_n^2\Theta_n + EI\Theta_n'' + kGA(V_n' - \Theta_n) = 0 \tag{7}$$

$$GJ\Psi_n'' + I_s\omega_n^2\Psi_n - \omega_n^2\mu y_c V_n - E\Gamma\Psi_n'''' = 0 \tag{8}$$

$$kGA(\Theta_n' - V_n'') - \mu\omega_n^2 V_n + \mu y_c\omega_n^2\Psi_n = 0 \tag{9}$$

Eqs. (7)–(9) can be combined into one equation by either eliminating all but one of the three variables V_n, Θ_n and Ψ_n to give the following eighth-order differential equation

$$\begin{aligned}\{d\,\mathrm{D}^8 + (b_n d(s+r) - 1)\,\mathrm{D}^6 - (b_n(s+r+d-b_n srd) + a_n)\,\mathrm{D}^4 - b_n(a_n r + b_n sr - 1 + a_n cs)\,\mathrm{D}^2 \\ + a_n c b_n(1 - b_n rs)\}X_n = 0\end{aligned} \tag{10}$$

where

$$X_n = V_n,\ \Theta_n \text{ or } \Psi_n,\quad \mathrm{D} = \mathrm{d}/\mathrm{d}\xi,\quad \xi = x/L$$
$$a_n = I_s\omega_n^2 L^2/GJ,\quad b_n = \mu\omega_n^2 L^4/EI,\quad c = 1 - \mu y_c^2/I_s,\quad d = E\Gamma/GJL^2$$
$$r = I/AL^2,\quad s = EI/kAGL^2$$

Note that d, r and s describe the effects of warping stiffness, rotary inertia and shear deformation, respectively. Any one of these parameters can be set to zero so that the corresponding effect can be optionally ignored.

The solution of the differential equation (10) can be obtained by substituting the trial solution $X_n = e^{\kappa_n \xi}$ to give the characteristic equation

$$d\kappa_n^8 + (b_n d(s+r) - 1)\kappa_n^6 - (b_n(s+r+d-b_n srd) + a_n)\kappa_n^4 - b_n(a_n r + b_n sr - 1 + a_n cs)\kappa_n^2 + a_n cb_n(1 - b_n rs) = 0 \tag{11}$$

Let

$$\chi_n = \kappa_n^2 \tag{12}$$

Substituting Eq. (12) into Eq. (11) gives

$$d\chi_n^4 + (b_n d(s+r) - 1)\chi_n^3 - (b_n(s+r+d-b_n srd) + a_n)\chi_n^2 - b_n(a_n r + b_n sr - 1 + a_n cs)\chi_n + a_n cb_n(1 - b_n rs) = 0 \tag{13}$$

It has been found from the numerical computation that, within the practical range, all four roots of Eq. (13) are real, two of them negative and the other two positive. Suppose that the four roots are $\chi_{n1}, \chi_{n2}, -\chi_{n3}, -\chi_{n4}$, where χ_{nj} $(j = 1-4)$ are real and positive. Then the eight roots of the characteristic equation (11) are

$$\alpha_n, -\alpha_n, \beta_n, -\beta_n, i\gamma_n, -i\gamma_n, i\delta_n, -i\delta_n$$

where $i = \sqrt{-1}$ and $\alpha_n = \sqrt{\chi_{n1}}$, $\beta_n = \sqrt{\chi_{n2}}$, $\gamma_n = \sqrt{\chi_{n3}}$, $\delta_n = \sqrt{\chi_{n4}}$.

It follows that the solution of Eq. (10) is of the following forms

$$V_n(\xi) = c_1^* \cosh\alpha_n\xi + c_2^* \sinh\alpha_n\xi + c_3^* \cosh\beta_n\xi + c_4^* \sinh\beta_n\xi + c_5^* \cos\gamma_n\xi + c_6^* \sin\gamma_n\xi + c_7^* \cos\delta_n\xi + c_8^* \sin\delta_n\xi \tag{14}$$

$$\Psi_n(\xi) = t_{n1}c_1^* \cosh\alpha_n\xi + t_{n1}c_2^* \sinh\alpha_n\xi + t_{n2}c_3^* \cosh\beta_n\xi + t_{n2}c_4^* \sinh\beta_n\xi + t_{n3}c_5^* \cos\gamma_n\xi + t_{n3}c_6^* \sin\gamma_n\xi + t_{n4}c_7^* \cos\delta_n\xi + t_{n4}c_8^* \sin\delta_n\xi \tag{15}$$

$$\Theta_n(\xi) = t_{n5}c_2^* \cosh\alpha_n\xi + t_{n5}c_1^* \sinh\alpha_n\xi + t_{n6}c_4^* \cosh\beta_n\xi + t_{n6}c_3^* \sinh\beta_n\xi + t_{n7}c_6^* \cos\gamma_n\xi - t_{n7}c_5^* \sin\gamma_n\xi + t_{n8}c_8^* \cos\delta_n\xi - t_{n8}c_7^* \sin\delta_n\xi \tag{16}$$

where c_1^*–c_8^* is a set of constants which can be determined from the boundary conditions, and

$$t_{n1} = a_n(1-c)b_n/(a_n b_n + b_n\alpha_n^2 - b_n d\alpha_n^4)y_c, \quad t_{n2} = a_n(1-c)b_n/(a_n b_n + b_n\beta_n^2 - b_n d\beta_n^4)y_c$$
$$t_{n3} = a_n(1-c)b_n/(a_n b_n - b_n\gamma_n^2 - b_n d\gamma_n^4)y_c, \quad t_{n4} = a_n(1-c)b_n/(a_n b_n - b_n\delta_n^2 - b_n d\delta_n^4)y_c$$
$$t_{n5} = \alpha_n/L(1 - b_n rs - \alpha_n^2 s), \quad t_{n6} = \beta_n/L(1 - b_n rs - \beta_n^2 s)$$
$$t_{n7} = \gamma_n/L(1 - b_n rs + \gamma_n^2 s), \quad t_{n8} = \delta_n/L(1 - b_n rs + \delta_n^2 s)$$

The following boundary conditions of the thin-walled beams are considered:

Clamped edge: $V_n = 0$, $\Psi_n = 0$, $\Theta_n = 0$, $\Psi_n' = 0$;
Free edge: $V_n' - \Theta_n = 0$, $d\Psi_n''' - \Psi_n' = 0$, $\Theta_n' = 0$, $\Psi_n'' = 0$.

For the clamped-free beams, applying the above boundary conditions to Eqs. (14)–(16) at $\xi = 0$ and 1 obtains a set of eight homogeneous algebraic equations

$$[\Pi]\{c^*\} = 0 \tag{17}$$

where $[\Pi]$ is a 8×8 matrix specified by the boundary conditions and $\{c^*\}$ is a 8×1 vector of unknown constants $c_1^*, c_2^*, \ldots, c_8^*$. Eq. (17) has nontrivial solutions for $\{c^*\}$ when the determinant of $[\Pi]$ vanishes; that is,

$$\det[\Pi] = 0 \tag{18}$$

or, more precisely, when the rank of $[\Pi]$ is less than eight. Together, Eqs. (18) and (11) must be solved numerically for the eigenvalues of the given modes; once they are known, the mode shapes are specified by Eq. (17).

In general, the solutions must be obtained iteratively. A value is chosen for ω_n, then Eqs. (11) and (12) are solved for the corresponding κ_n and χ_n. The roots χ_{nj} along with ω_n are used to compute the rank of the matrix $[\Pi]$, by calculating the value of its determinant, for example. If Eq. (18) is not satisfied to within some tolerance, then the value of ω_n must be changed and the process is repeated.

The simplest scheme for determining the natural frequencies is to specify a starting value for ω_n and an increment $\Delta\omega_n$ and then simply march up the frequency until all of the desired natural frequencies have been obtained. Direct computation of the determinant is very cumbersome even for moderately large matrices. Although more efficient algorithms exist for calculating the determinant, most require that the matrix be nonsingular and are therefore not useful. A much better approach is to determine the rank deficiency by using an alternate technique such as singular value decomposition.

Singular value decomposition, which is not restricted to square matrices, decomposes a matrix $[\Pi]$ into two orthonormal matrices $[P]$ and $[Q]$ and a diagonal matrix $[D]$ (Bay, 1999) in the form

$$[\Pi] = [P][D][Q]^{\mathrm{T}} \tag{19}$$

The diagonal elements that consist of $[D]$ are called the singular values and the number of nonzero singular values corresponds to the rank of $[\Pi]$. The values of ω_n for which one of the singular values goes to zero are the natural frequencies. As usual, because $[\Pi]$ is singular, Eq. (17) can only be used to calculate seven of the eight unknowns c_j^* in terms of the remaining one.

Based on Eqs. (7)–(9) and the boundary conditions, the following orthogonality for different mode shapes of the thin-walled Timoshenko beams can be derived as

$$\int_0^1 \{(\rho I \Theta_m \Theta_n + \mu V_m V_n + I_{\mathrm{s}} \Psi_m \Psi_n) - \mu y_{\mathrm{c}}(V_m \Psi_n + V_n \Psi_m)\}\,\mathrm{d}\xi = \overline{m}_n \delta_{mn} \tag{20}$$

where $\overline{m}_n$ is the generalized mass in the nth mode, δ_{mn} is the Kronecker delta function.

With the free vibration natural frequencies, mode shapes, and orthogonality condition described above, it is now possible to investigate the general random vibration problem of the damped thin-walled Timoshenko beams.

3. Random vibration analysis of the thin-walled Timoshenko beams

For forced vibration of the thin-walled Timoshenko beams, assume $v(x,t)$, $\theta(x,t)$, $\psi(x,t)$ can be expanded in terms of the eigenfunctions to give the following three equations

$$v(x,t) = v(\xi L, t) = \sum_{n=1}^{\infty} q_n(t) V_n(\xi) \tag{21}$$

$$\psi(x,t) = \psi(\xi L, t) = \sum_{n=1}^{\infty} q_n(t) \Psi_n(\xi) \tag{22}$$

$$\theta(x,t) = \theta(\xi L, t) = \sum_{n=1}^{\infty} q_n(t)\Theta_n(\xi) \tag{23}$$

where $q_n(t)$ are the generalized time-dependent coordinates for each mode. Substituting Eqs. (21)–(23) into Eqs. (1)–(3) and using Eqs. (7)–(9) yields

$$\sum_{n=1}^{\infty}[\mu(V_n - y_c\Psi_n)\ddot{q}_n + c_1(V_n - y_c\Psi_n)\dot{q}_n + \mu\omega_n^2(V_n - y_c\Psi_n)q_n] = f(\xi,t) \tag{24}$$

$$\sum_{n=1}^{\infty}[\rho I\Theta_n\ddot{q}_n + c_3\Theta_n\dot{q}_n + \rho I\omega_n^2\Theta_n q_n] = 0 \tag{25}$$

$$\sum_{n=1}^{\infty}[(I_s\Psi_n - \mu y_c V_n)\ddot{q}_n + (c_2\Psi_n - c_1 V_n y_c)\dot{q}_n + \omega_n^2(I_s\Psi_n - \mu y_c V_n)q_n] = m(\xi,t) \tag{26}$$

where superscript dot denotes derivative with respect to time.

Multiplying Eqs. (24)–(26) by V_m, Θ_m and Ψ_m respectively, then summing up these three equations and integrating from 0 to 1, and using orthogonality condition (20) gives

$$\ddot{q}_n(t) + 2\zeta_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = [F_n(t) + M_n(t)] \tag{27}$$

where

$$F_n(t) = \frac{1}{\overline{m}_n}\int_0^1 V_n(\xi)f(\xi,t)\,\mathrm{d}\xi, \quad M_n(t) = \frac{1}{\overline{m}_n}\int_0^1 \Psi_n(\xi)m(\xi,t)\,\mathrm{d}\xi$$

ζ_n is a nondimensional quantity known as the viscous damping factor. Here the following assumption has been made

$$\int_0^1 [(c_1 V_m V_n + c_2\Psi_m\Psi_n + c_3\Theta_m\Theta_n) - c_1 y_c(V_m\Psi_n + V_n\Psi_m)]\,\mathrm{d}\xi = 2\zeta_n\omega_n\overline{m}_n\delta_{mn}$$

The dynamic response of the thin-walled Timoshenko beams subjected to stationary, ergodic random excitations with zero initial conditions is investigated in the frequency domain by using the frequency response function method.

From Eq. (27), the cross-spectral density function $S_{q_nq_l}(\Omega)$ of the generalized time-dependent coordinate $q_n(t)$ can be derived as

$$S_{q_nq_l}(\Omega) = H_n^*(\Omega)[S_{F_nF_l}(\Omega) + S_{M_nM_l}(\Omega)]H_l(\Omega) \tag{28}$$

where $H_l(\Omega)$ is the frequency response function

$$H_l(\Omega) = \frac{1}{(\omega_l^2 - \Omega^2 + 2i\zeta_l\Omega\omega_l)}$$

$H_n^*(\Omega)$ is the complex conjugate of $H_n(\Omega)$, $S_{F_nF_l}(\Omega)$ is the cross-spectral density function between $F_n(t)$ and $F_l(t)$, $S_{M_nM_l}(\Omega)$ is the cross-spectral density function between $M_n(t)$ and $M_l(t)$. Since it is assumed that the random excitations $f(\xi,t)$ and $m(\xi,t)$ are stationary in time, then so are the generalized forces $F_n(t)$ and $M_n(t)$. Furthermore, $F_n(t)$ and $M_n(t)$ are assumed to be independent random processes so that the cross-spectral density function between $F_n(t)$ and $M_n(t)$ can be excluded.

Based on the expressions of the generalized forces $F_n(t)$ and $M_n(t)$, the cross-spectral density functions $S_{F_nF_l}(\Omega)$ and $S_{M_nM_l}(\Omega)$ can be obtained explicitly as, respectively

$$\begin{aligned} S_{F_nF_l}(\Omega) &= \frac{1}{\overline{m}_n\overline{m}_l}\int_0^1\int_0^1 V_n(\xi_1)V_l(\xi_2)S_f(\xi_1,\xi_2,\Omega)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ S_{M_nM_l}(\Omega) &= \frac{1}{\overline{m}_n\overline{m}_l}\int_0^1\int_0^1 \Psi_n(\xi_1)\Psi_l(\xi_2)S_m(\xi_1,\xi_2,\Omega)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \end{aligned} \tag{29}$$

where $S_f(\xi_1,\xi_2,\Omega)$ is the distributed cross-spectral density function between the bending excitations $f(\xi_1,t)$ and $f(\xi_2,t)$, $S_m(\xi_1,\xi_2,\Omega)$ is the distributed cross-spectral density function between the torsional excitations $m(\xi_1,t)$ and $m(\xi_2,t)$.According to Eqs. (21)–(23), with the help of Eqs. (28) and (29), the cross-spectral density functions $S_v(\xi_1,\xi_2,\Omega)$, $S_\psi(\xi_1,\xi_2,\Omega)$ and $S_\theta(\xi_1,\xi_2,\Omega)$ of the bending translation $v(\xi,t)$, torsional rotation $\psi(\xi,t)$ and bending rotation $\theta(\xi,t)$ can be written as

$$S_v(\xi_1,\xi_2,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)V_n(\xi_1)V_l(\xi_2) \tag{30}$$

$$S_\psi(\xi_1,\xi_2,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Psi_n(\xi_1)\Psi_l(\xi_2) \tag{31}$$

$$S_\theta(\xi_1,\xi_2,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Theta_n(\xi_1)\Theta_l(\xi_2) \tag{32}$$

where $h_n^*(\Omega)$ is the complex conjugate of $h_n(\Omega)$

$$h_l(\Omega) = \frac{1}{\overline{m}_l(\omega_l^2 - \Omega^2 + 2\mathrm{i}\zeta_l\omega_l\Omega)}$$

$$\eta_{nl}(\Omega) = \int_0^1\int_0^1 \{V_n(\xi_1)V_l(\xi_2)S_f(\xi_1,\xi_2,\Omega) + \Psi_n(\xi_1)\Psi_l(\xi_2)S_m(\xi_1,\xi_2,\Omega)\}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

For $\xi_1=\xi_2=\xi$, the cross-spectral density functions $S_v(\xi_1,\xi_2,\Omega)$, $S_\psi(\xi_1,\xi_2,\Omega)$ and $S_\theta(\xi_1,\xi_2,\Omega)$ reduce to the spectral density functions $S_v(\xi,\Omega)$, $S_\psi(\xi,\Omega)$ and $S_\theta(\xi,\Omega)$

$$S_v(\xi,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)V_n(\xi)V_l(\xi) \tag{33}$$

$$S_\psi(\xi,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Psi_n(\xi)\Psi_l(\xi) \tag{34}$$

$$S_\theta(\xi,\Omega) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} h_n^*(\Omega)h_l(\Omega)\eta_{nl}(\Omega)\Theta_n(\xi)\Theta_l(\xi) \tag{35}$$

The mean square values of the bending translation, torsional rotation and bending rotation can be found by integrating the corresponding spectral density functions over all frequencies

$$\langle v^2(\xi,t)\rangle = \int_{-\infty}^{\infty} S_v(\xi,\Omega)\,\mathrm{d}\Omega \tag{36}$$

$$\langle\psi^2(\xi,t)\rangle = \int_{-\infty}^{\infty} S_\psi(\xi,\Omega)\,\mathrm{d}\Omega \tag{37}$$

$$\langle\theta^2(\xi,t)\rangle = \int_{-\infty}^{\infty} S_\theta(\xi,\Omega)\,\mathrm{d}\Omega \tag{38}$$

If the external random excitations are assumed to follow the Gaussian probability distribution, the response probability will also be Gaussian, and therefore the response can be fully described by its spectral density function.

For simplicity, suppose that there is only one randomly varying concentrated bending excitation acting on the thin-walled beam at $\xi = \xi_f$. In this case, $\eta_{nl}(\Omega)$ in Eqs. (30)–(32) can be simplified as

$$\eta_{nl}(\Omega) = V_n(\xi_f)V_l(\xi_f)S_f(\Omega) \tag{39}$$

The spectral density functions of the bending translation, torsional rotation and bending rotation are then given by Eqs. (33)–(35) as

$$S_v(\xi,\Omega) = \left|\sum_{l=1}^{\infty} h_l(\Omega)V_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega) \tag{40}$$

$$S_\psi(\xi,\Omega) = \left|\sum_{l=1}^{\infty} h_l(\Omega)\Psi_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega) \tag{41}$$

$$S_\theta(\xi,\Omega) = \left|\sum_{l=1}^{\infty} h_l(\Omega)\Theta_l(\xi)V_l(\xi_f)\right|^2 S_f(\Omega) \tag{42}$$

4. Numerical results

Some numerical results are given to demonstrate the theoretical formulations derived in preceding sections, which can be directly applied to compute the random response of the thin-walled Timoshenko beams subjected to concentrated or distributed random excitations.

The first example is a cantilever thin-walled beam with monosymmetrical semi-circular cross-section, shown in Fig. 2. The geometrical and physical properties of the thin-walled beam are given as follows:

$$I = 9.26\times10^{-8}\ \mathrm{m}^4,\quad J = 1.64\times10^{-9}\ \mathrm{m}^4,\quad I_\mathrm{s} = 0.000501\ \mathrm{kg\,m},\quad y_\mathrm{c} = 0.0155\ \mathrm{m},\quad L = 0.82\ \mathrm{m}$$
$$\Gamma = 1.52\times10^{-12}\ \mathrm{m}^6,\quad \mu = 0.835\ \mathrm{kg\,m}^{-1}\quad E = 68.9\times10^9\ \mathrm{N\,m}^{-2},\quad G = 26.5\times10^9\ \mathrm{N\,m}^{-2}$$
$$k = 0.5,\quad A = 3.08\times10^{-4}\ \mathrm{m}^2,\quad \rho = 2711.04\ \mathrm{kg\,m}^{-3}$$

To validate and confirm the accuracy of the present numerical results, the natural frequencies and mode shapes of the above thin-walled beam for undamped free vibration are computed first. The first five natural frequencies of the thin-walled beam are shown in Table 1. The corresponding first five normal mode shapes including the shear deformation, rotary inertia and warping stiffness are shown in Fig. 3(a)–(e). It can be seen from Table 1 that the agreement between the present results and those of Bercin and Tanaka (1997) is very excellent. As expected, the corresponding mode shapes of Fig. 3 also resemble the ones given by Bercin

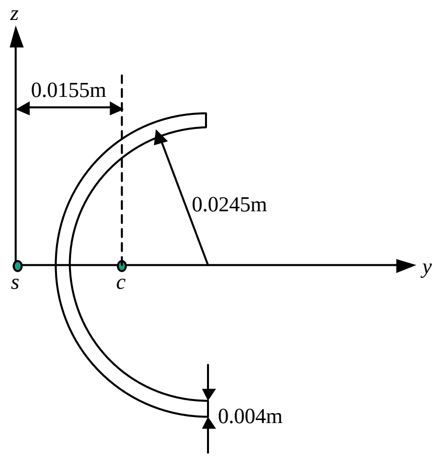

Fig. 2. Beam cross-section used in numerical example 1.

Table 1
Coupled bending–torsional natural frequencies of the cantilever semi-circular section beam

Frequency order	Natural frequency (Hz)			
	Only warping ignored	Only shear deformation and rotary inertia ignored	Shear deformation, rotary inertia and warping included	
			Present results	Results in Bercin and Tanaka (1997)
1	62.34	63.79	63.50	63.51
2	129.87	137.68	137.38	137.39
3	259.21	278.35	275.81	275.82
4	418.89	484.77	481.09	481.10
5	605.21	663.84	639.75	639.76

and Tanaka (1997) very closely. Also, to be consistent with Bercin and Tanaka (1997), the bending rotation and torsional rotation are multiplied by the distance y_c when plotting the mode shapes.

Based on the natural frequencies and mode shapes of the thin-walled beam, the mean square values of bending translation, bending rotation and torsional rotation due to a random varying concentrated bending excitation can be computed without any difficulty. The random bending excitation is assumed to be an ideal white noise, so the $S_f(\omega)$ in Eqs. (40)–(42) can be replaced by a constant, i.e. $S_f(\omega) = S_0$ (S_0 is a constant). In Figs. 4–6, respectively, are shown the mean square values of bending translation, bending rotation and torsional rotation along the length of the cantilever thin-walled beam subjected to an ideal white noise concentrated bending excitation acting at the tip of the beam. The value of the damping coefficient has been taken as 0.01. The mean square bending displacements and torsional displacement accounting for the shear deformation and rotary inertia have a little difference from the ones excluding the shear deformation and rotary inertia. But the mean square bending displacements and torsional displacement including the warping stiffness are significantly different from the ones excluding the warping stiffness, as can be seen from Figs. 4–6. The numerical results show that it is necessary to consider the

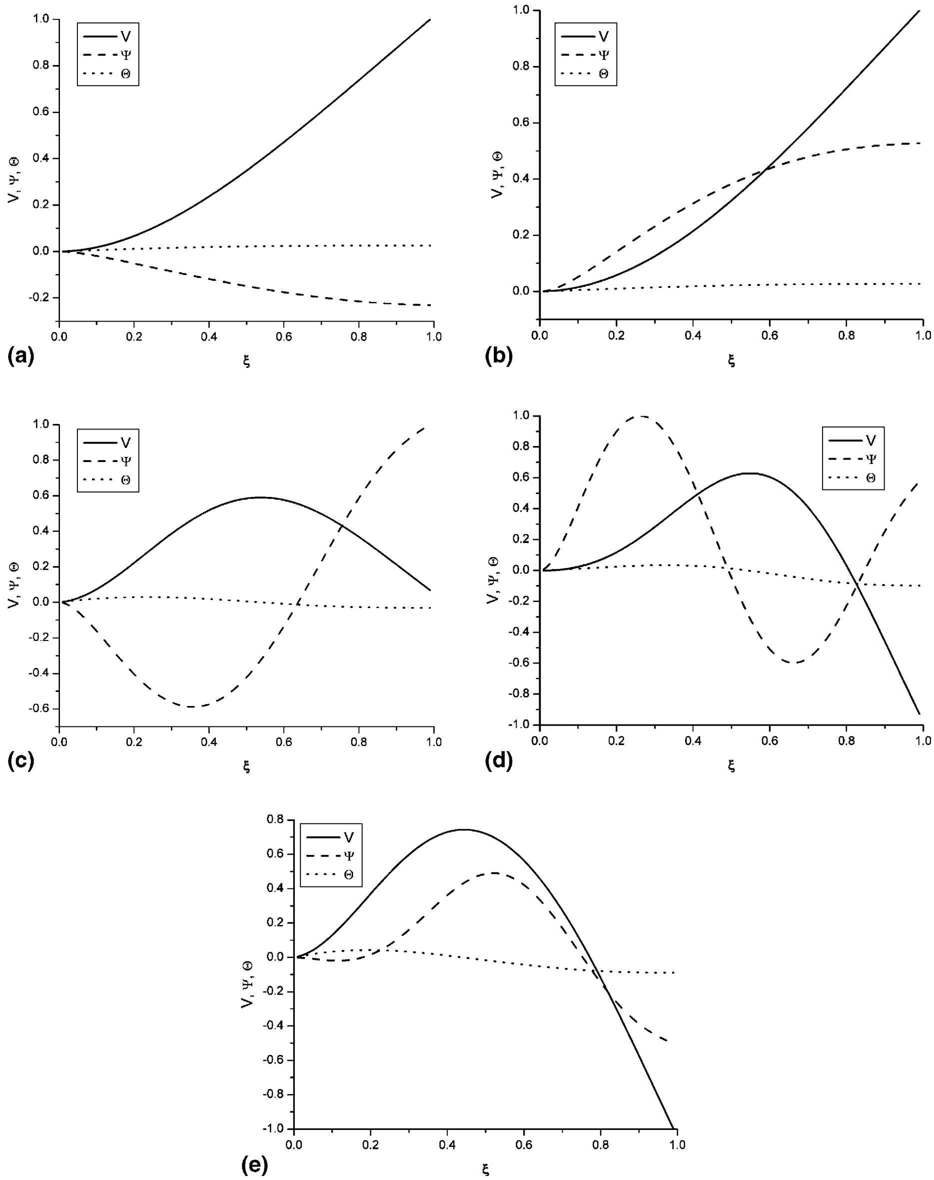

Fig. 3. The first five normal mode shapes of example 1 with the warping stiffness, shear deformation and rotary inertia included (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

warping stiffness effect when the mean square displacements of this thin-walled beam are calculated. The effects of shear deformation and rotary inertia on the mean square displacements seem to be insignificant for this specific problem investigated.

A cantilever thin-walled uniform beam with a monosymmetrical channel cross-section is considered next, shown in Fig. 7. The geometrical properties and physical properties of the beam are given below:

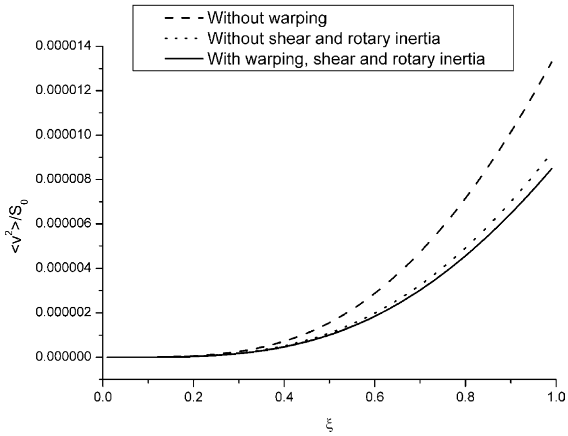

Fig. 4. Mean square bending translation along the length of the cantilever thin-walled beam.

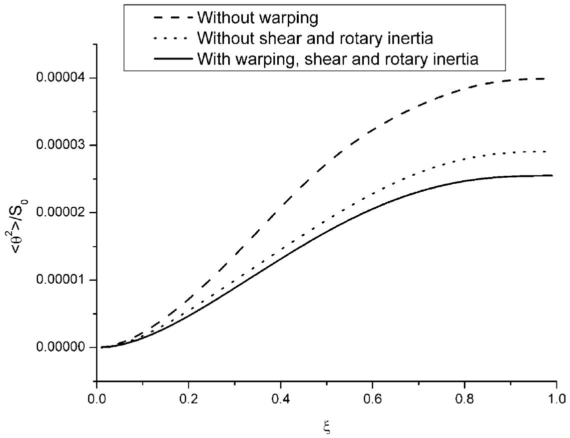

Fig. 5. Mean square bending rotation along the length of the cantilever thin-walled beam. the length of the cantilever thin-walled beam.

$$I = 1.449 \times 10^{-3}\ \text{m}^4,\quad J = 1.223 \times 10^{-5}\ \text{m}^4,\quad I_s = 56.87\ \text{kg m},\quad y_c = 0.336\ \text{m},\quad L = 3.2\ \text{m}$$
$$\Gamma = 3.885 \times 10^{-5}\ \text{m}^6,\quad \mu = 225\ \text{kg m}^{-1},\quad E = 2.1 \times 10^{11}\ \text{N m}^{-2},\quad G = 8 \times 10^{10}\ \text{N m}^{-2}$$
$$k = 0.5136,\quad A = 0.012856\ \text{m}^2,\quad \rho = 17501.6\ \text{kg m}^{-3}$$

The first five natural frequencies of the cantilever thin-walled beam with and without inclusion of the warping stiffness and/or shear deformation and rotary inertia are calculated and the numerical results are shown in Table 2, along with the natural frequencies of Bercin and Tanaka (1997). The corresponding mode shapes of the first five normal modes including the warping stiffness, shear deformation and rotary inertia are plotted in Fig. 8(a)–(e). Again, it can be seen from Table 2 and Fig. 8, the natural frequencies and mode shapes obtained from the present theory completely agree with those given by Bercin and Tanaka (1997).

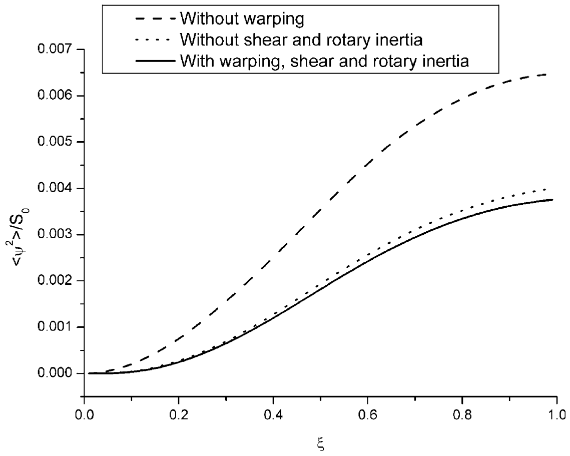

Fig. 6. Mean square torsional rotation along the length of the cantilever thin-walled beam.

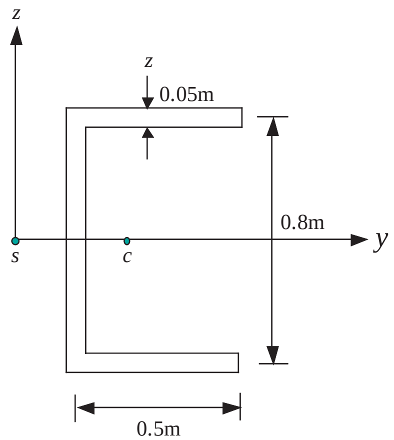

Fig. 7. Beam cross-section used in numerical example 2.

Table 2
Coupled bending–torsional natural frequencies of the cantilever channel section beam

Frequency order	Natural frequency (Hz)			
	Only warping ignored	Only shear deformation and rotary inertia ignored	Shear deformation, rotary inertia and warping included	
			Present results	Results in Bercin and Tanaka (1997)
1	10.17	24.02	23.78	23.79
2	30.51	88.53	77.24	78.26
3	51.08	131.40	124.77	124.78
4	71.29	358.57	295.25	295.26
5	74.94	549.81	334.87	334.88

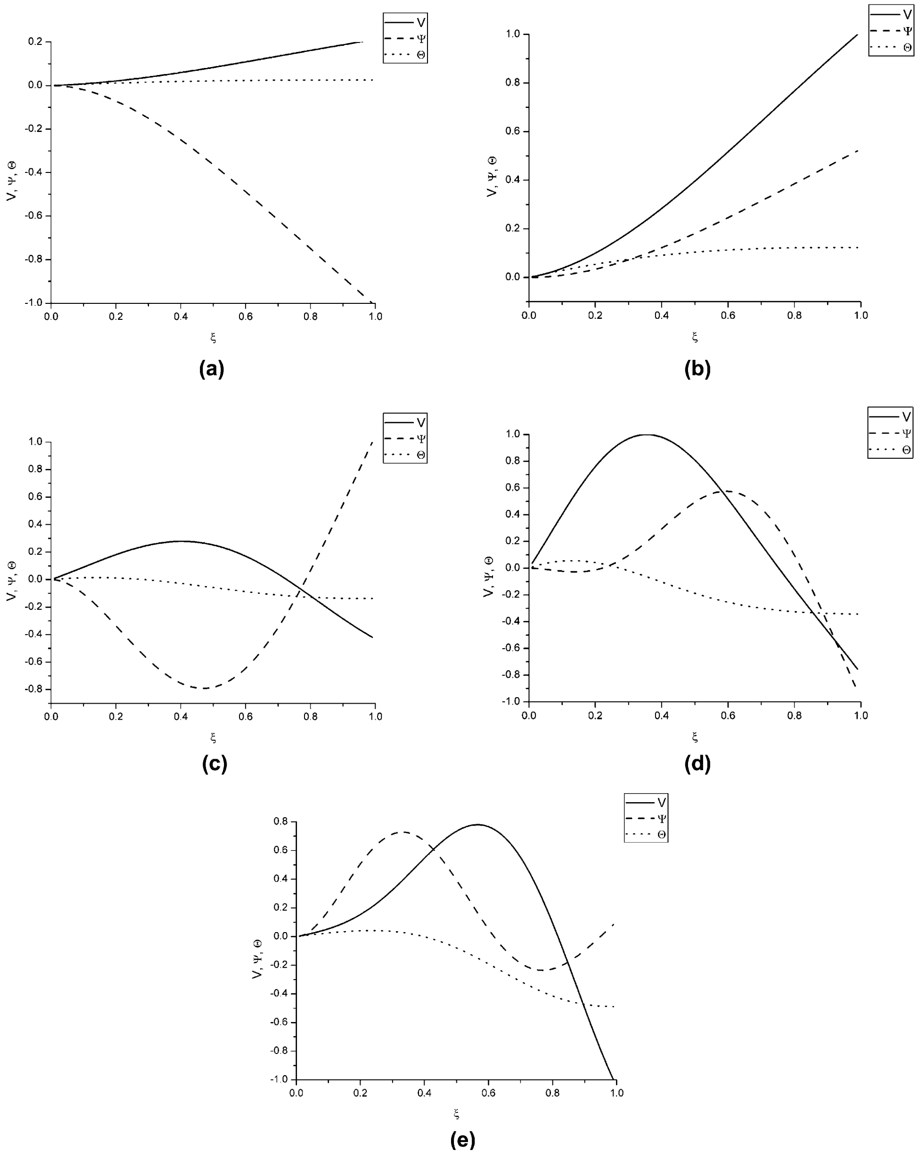

Fig. 8. The first five normal mode shapes of example 2 with the warping stiffness, shear deformation and rotary inertia included (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5.

Following the same procedure discussed above, the random response can be computed without any difficulty based on the natural frequencies and mode shapes. To compare the results obtained from the present theory including the warping stiffness, shear deformation and rotary inertia with those given by the theory excluding the warping stiffness or shear deformation and rotary inertia, the mean square values of the bending translation, bending rotation and torsional rotation due to a random varying concentrated bending excitation are calculated. The value of the damping coefficient used in computation is 0.01. In Figs. 9–11, respectively, are shown the mean square values of the bending translation, bending rotation and

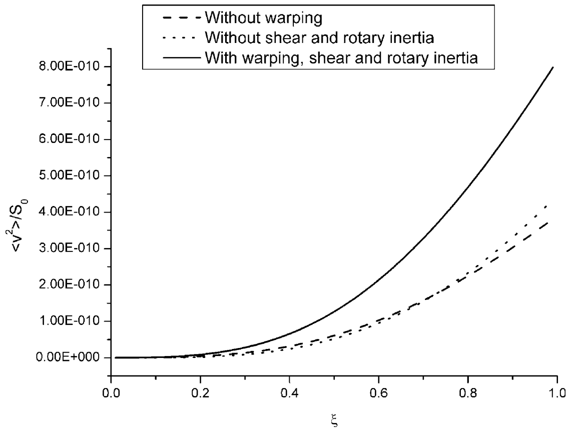

Fig. 9. Mean square bending translation along the length of the cantilever thin-walled beam.

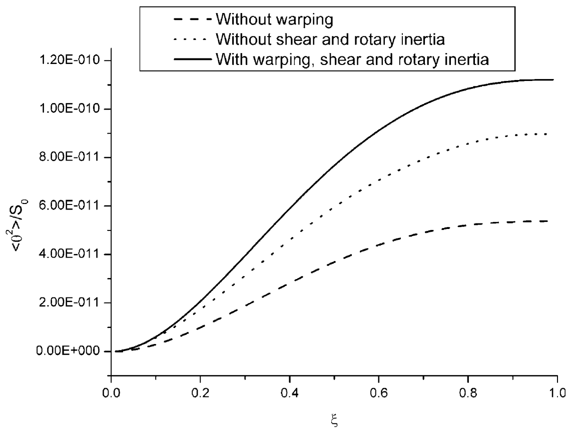

Fig. 10. Mean square bending rotation along the length of the cantilever thin-walled beam. the length of the cantilever thin-walled beam.

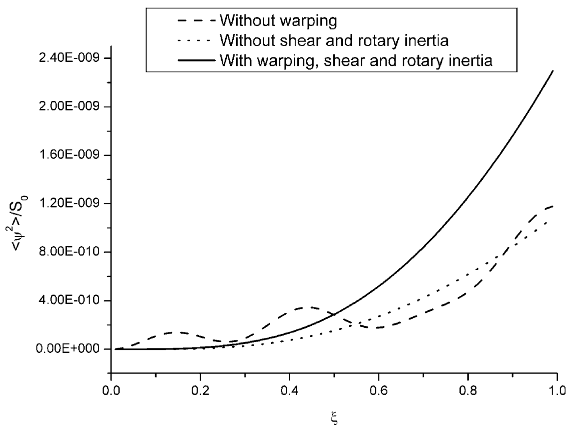

Fig. 11. Mean square torsional rotation along the length of the cantilever thin-walled beam.

Table 3
Mean square values of the bending and torsional response at the tip of the cantilever thin-walled beam

	Warping ignored	Shear deformation and rotary inertia ignored	Present theory
Bending translation $\langle v^2\rangle/S_0$	3.82×10^{-10}	4.35×10^{-10}	7.98×10^{-10}
Bending rotation $\langle \theta^2\rangle/S_0$	5.38×10^{-11}	8.97×10^{-11}	1.12×10^{-10}
Torsional rotation $\langle \psi^2\rangle/S_0$	1.18×10^{-9}	1.09×10^{-9}	2.30×10^{-9}

torsional rotation along the length of the thin-walled beam subjected to an ideal white noise concentrated bending excitation acting at the tip of the beam. As can be seen from Figs. 9–11, the mean square values of the bending displacements and torsional displacement predicted by the theory considering the warping stiffness, shear deformation and rotary inertia are significantly different from those obtained from the theory excluding the warping stiffness or shear deformation and rotary inertia. The difference is more pronounced at the tip of the cantilever thin-walled beam. The mean square values of the bending displacements and torsional displacement at the tip of the cantilever thin-walled beam are shown in Table 3. The numerical results illustrate quite well that the warping stiffness and shear deformation and rotary inertia can have strong influences on the random response of the thin-walled beam. So it is absolutely necessary to include the warping stiffness, shear deformation and rotary inertia when the mean square displacements of this channel section thin-walled beam are computed.

5. Conclusions

An analytical method for determining the bending–torsion coupled random response of the thin-walled beams with monosymmetrical cross-sections is developed. The method takes into account the effects of warping stiffness, shear deformation and rotatory inertia. The external random excitations can be concentrated or distributed along the beam length and are assumed to be stationary and ergodic. The mean square displacements of the thin-walled beams are computed by using the normal mode method combined with frequency response function method. The effects of warping stiffness, shear deformation and rotary inertia on the random response of two appropriately chosen thin-walled beams from the literature are demonstrated and discussed.

Acknowledgements

The authors would like to thank the reviewers for their helpful and important comments and suggestions.

Appendix A

The damped governing differential equations for the bending–torsion coupled forced vibration of the thin-walled Timoshenko beams can be derived using the Hamilton's principle as follows.

The total strain energy U of a thin-walled Timoshenko beam shown in Fig. 1 is given by

$$U = \frac{1}{2}\int_0^L \left\{ EI(\theta')^2 + kAG(v'-\theta)^2 + E\Gamma(\psi'')^2 + GJ(\psi')^2 \right\} \mathrm{d}x \tag{A.1}$$

where all the variables and symbols are defined in Section 2.

The total kinetic energy T of a thin-walled Timoshenko beam is given by

$$T = \frac{1}{2}\int_0^L \left[\mu(\dot{v}^2 - 2y_c\dot{v}\dot{\psi}) + I_s\dot{\psi}^2 + \rho I\dot{\theta}^2\right] dx \tag{A.2}$$

The governing equations of motion and the boundary conditions can be derived conveniently by means of the Hamilton's principle, which can be stated in the form

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\, dt = 0 \tag{A.3}$$

$$\delta v = \delta\theta = \delta\psi = \delta\psi' = 0 \quad \text{at } t = t_1, t_2$$

Herein T is the kinetic energy, U the potential energy, δW the virtual work of the nonconservative forces, which can be written as

$$\delta W = \int_0^L \left[f(x,t)\,\delta v + m(x,t)\,\delta\psi - c_1(\dot{v} - y_c\dot{\psi})\,\delta v - (c_2\dot{\psi} - c_1 y_c\dot{v})\,\delta\psi - c_3\dot{\theta}\,\delta\theta\right] dx \tag{A.4}$$

Substituting Eqs. (A.1), (A.2) and (A.4) into Eq. (A.3) and carrying out the usual steps yields the governing equations of motion and the boundary conditions.

(a) The governing equations of motion

$$\mu\ddot{v} - \mu y_c\ddot{\psi} - kAGv'' + kAG\theta' + c_1(\dot{v} - y_c\dot{\psi}) = f(x,t) \tag{A.5}$$

$$I_s\ddot{\psi} - \mu y_c\ddot{v} + E\Gamma\psi'''' - GJ\psi'' + (c_2\dot{\psi} - c_1 y_c\dot{v}) = m(x,t) \tag{A.6}$$

$$\rho I\ddot{\theta} - EI\theta'' - kAGv' + kAG\theta + c_3\dot{\theta} = 0 \tag{A.7}$$

(b) The boundary conditions at the ends ($x = 0, L$)

$$(-kAGv' + kAG\theta)\,\delta v = 0 \tag{A.8}$$

$$(E\Gamma\psi''' - GJ\psi')\,\delta\psi = 0 \tag{A.9}$$

$$(-EI\theta')\,\delta\theta = 0 \tag{A.10}$$

$$(-E\Gamma\psi'')\,\delta\psi' = 0 \tag{A.11}$$

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